Annual Status Report

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Theoretical Investigation of the Force-and Dynamically Coupled Torsional-Axial-Lateral Dynamic Response of Geared Rotors

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Introduction

Because of the magnitude of the accomplishments of the last 6 months, it was necessary to deviate from the usual letter-type report. The major accomplishments are:

- Resolution of the difficulties in solution methodology to be used to deal with the potentially highly nonlinear rotor equations when dynamic coupling is included.
- A solution methodology is selected to solve the nonlinear differential equations. The selected method was verified to give good results even at large nonlinearity levels.
- Extension of the transfer matrix methodology to the solution of nonlinear problems. It is believed that this is a technical first. Transfer matrix methods are obstensibly a linear mechanics method. They now have nonlinear capabilities. Moreover, these nonlinear analysis capabilities extend into the large nonlinear regime. It is hoped that this work will open up new applications of transfer matrices to structural mechanics problems.

These topics will be discussed in the above order in that this chronology presents the logical progression to the transfer matrix breakthrough.

RESOLUTION OF DIFFICULTIES WITH PREVIOUS SOLUTION METHODOLOGY

The solution technique as reported in the previous report was found to be deficient in several areas. Also, it was somewhat erroneously termed a harmonic-balance technique, while, in fact, it was more a modification of the perturbation technique. The greatest deficiency was that it produced accurate results only for cases where the nonlinearity was small, like the perturbation technique. Since our motivation for dealing with the dynamically coupled rotor problem is that it is suspected that the nonlinearities are not small. This was a serious shortcoming. Also, the approximate solution predicted regions of subharmonic instability that could not be substantiated by time transient numerical solutions. Hence, a superior technique was sought for solving problems of this type.

The technique investigated is indeed the classical harmonic-balance technique, but with a slightly different method of actually obtaining the solution. Indeed, the solution of linear ordinary differential equations with periodic solutions uses the harmonic-balance technique.

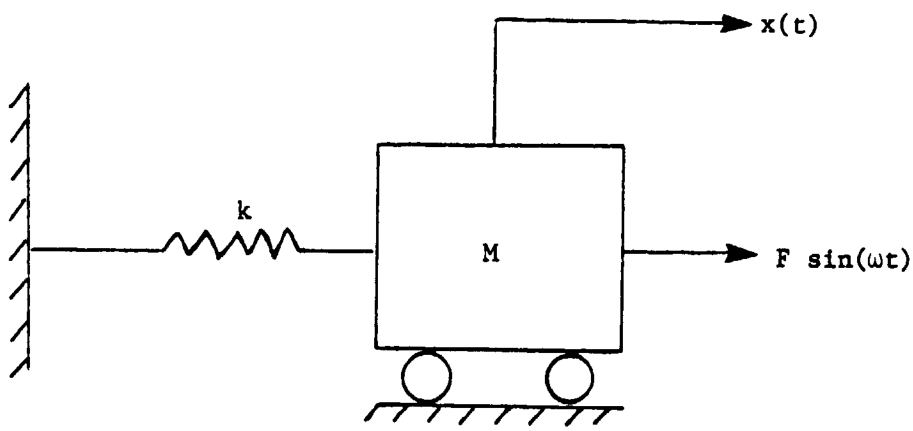
For a linear ordinary differential equation with a periodic solution, one assumes the form of the solution and substitutes this assumed solution into the differential equation. This results in an algebraic equation that can be solved for the unknown response amplitude and phase relationship. This results in an exact solution for the linear differential equation.

In dealing with nonlinear ordinary differential equations, this same technique can be used to obtain approximate periodic solutions. A form of the solution is assumed and substituted into the differential equation. Due to the nonlinearities, this often results in an algebraic equation containing trigonometric functions raised to integer powers. These terms can be reduced to simple harmonic functions through the use of trigonometric identities. The resulting equation is algebraic in nature, containing trigonometric terms, primarily sines and cosines of many frequencies on the right-hand side of the equation while containing only terms of a single frequency on the left-hand side. It can be argued that if the resulting equation is indeed an identity, the coefficients of the sines and cosines on both sides of the equation must form and identity. Hence, one equates coefficients of like frequencies of sines and cosines, ignoring those those harmonic terms not in the assumed solution, and obtains simple algebraic expressions which can be solved for the unknown amplitude and phase information. Perhaps at this point an example would serve to clarify the above description.

SOLUTION OF DUFFING'S EQUATION BY THE HARMONIC-BALANCE TECHNIQUE

Duffing's equation is representative of a system consisting of a mass resting on a spring with a cubic nonlinearity as shown in Fig. 1. The equation of motion is

$$m\ddot{x} + k(x+\epsilon x^3) = F \sin(\omega t).$$



$$F_{spring} = k(x + \varepsilon x^3)$$

Fig. 1 Harmonic Oscillator with Nonlinear Spring

This equation was chosen because it has been thoroughly investigated and the results of the harmonic-balance technique could be easily verified. The solution methodology is as follows:

From Newton's Laws, one can argue that one component of the response must be at the excitation frequency, so initially a solution of the form

$$x(t) = x_1 \sin(\omega t)$$

is assumed and substituted into the equation of motion. The resulting equation is

$$-m\omega^2 x_1 \sin(\omega t) + k(x_1 \sin(\omega t) + \varepsilon x_1^3 \sin^3(\omega t)) = F \sin(\omega t).$$

The term $\sin^3(\omega t)$ is simplified using the trigonometric identity as

$$\sin^3(\omega t) = 1/4(3 \sin(\omega t) - \sin(3\omega t)).$$

which yields the following equation of motion

$$-\omega^2 x_1 \sin(\omega t) + k/m\ x_1 \sin(\omega t) + \varepsilon k/4m\ x_1^3 (3 \sin(\omega t) - \sin(3\omega t)) = F/m \sin(\omega t).$$

At this point, one could apply the principle of harmonic balance, which would yield

$$-\omega^2 x_1 + k/m\ x_1 + 3k/4m\ \ \varepsilon x_1^3 = F/m$$

This could be solved for the unknown amplitude x_1. In doing this, one ignores the $\sin(3\omega t)$ term. This assumption is the reason the solution is called approximate. Since this $\sin(3\omega t)$ term was ignored, a better solution can be obtained by, upon the next iteration, letting the initial assumed solution contain a term of this form. Thus one assumes

$$x(t) = x_1 \sin(\omega t) + x_3 \sin(3\omega t)$$

and substitute this into the differential equation. Upon substitution into the equation of motion thi. yields

$$-m\omega^2(x_1 \sin(\omega t) + 9x_3 \sin(3\omega t)) + k(x_1 \sin(\omega t) + x_3 \sin(3\omega t))$$

$$+ \varepsilon k((x_1 \sin(\omega t) + x_3 \sin(3\omega t))^3) = F \sin(\omega t).$$

After simplification and application of the harmonic-balance principle, the differential equation is reduced to two simultaneous equations in terms of the unknowns x_1 and x_3. These equations are (letting $\gamma^2 = \omega^2 m/k$) for coefficients of $\sin(\omega t)$

$$3/4\ \varepsilon x_1^3 - 3/4\ \varepsilon x_1^2 x_3 + 3/2\ \varepsilon x_1 x_3^2 + (1-\gamma^2)x_1 - F/k = 0.$$

and for coefficients of $\sin(3\omega t)$

$$-1/4\ \varepsilon x_1^3 + 3/4\ \varepsilon x_3^3 + 3/2\ \varepsilon x_1^2 x_3 + (1-9\gamma^2)x_3 = 0.$$

As can be seen, these equations are coupled and nonlinear, hence an explicit solution cannot be formed, except in the case of small nonlinearities. However, the real roots of these equations represent the approximate solution to the differential equation. The methodology for finding the roots to these equations will be outlined in a subsequent section. This process could be repeated, that is readjusting the assumed form of the solution to account for the harmonic terms that were ignored in the previous step, yielding a better answer. However, the algebraic equations become extremely complex and hence, the process is usually terminated after one or two iterations.

Damping can readily be included in the problem, rendering the equation of motion as

$$m\ddot{x} + c\dot{x} + k(x+\varepsilon x^3) = F \sin(\omega t).$$

The only modification to the solution process is in the assumed form of the solution, which in this case is

$$x(t) = x_{11} \sin(\omega t) + x_{12} \cos(\omega t) + x_{31} \sin(3\omega t) + x_{32} \cos(3\omega t).$$

This is preferred over a complex formulation so that all complex roots of the algebraic equations can be ignored. As before, this assumed form of the solution is substituted into the equation of motion, this yielding four algebraic equations in terms of the four unknowns (x_{11}, x_{12}, x_{31}, x_{32}). Since the algebra for this case is unwieldy, the computer program FMACUT(*) was used to perform the algebraic manipulations. As before, the final step in the solution is to find the appropriate solution to the resulting algebraic equations.

SOLUTION OF THE NONLINEAR ALGEBRAIC EQUATIONS

In essence, the harmonic-balance technique for obtaining approximate solutions to nonlinear ordinary differential equations reduces the solu-

*FMACUT is a derivative of the computer program FORMAC, which does algebraic manipulations instead of numerical calculations.

tions to finding the real roots of simultaneous, nonlinear algebraic equations. Once these several roots are found, a physical principle must be employed to select the correct one. Since the equations are nonlinear, there is no direct solution technique for finding the roots. Hence, one must resort to an iterative technique. The procedure used is based on an approximate solution and employs a truncated Taylor series to converge to a more precise answer. Again, perhaps the procedure is best described with the use of an example.

The algebraic equations resulting from the analysis of Duffing's equation in the previous section were:

$$f(x_1,x_3) = 3/4\ \varepsilon x_1^3 - 3/4\ \varepsilon x_1^2 x_3 + 3/2\ \varepsilon x_1 x_3^2 + (1-\gamma^2)x_1 - F/k = 0$$

$$g(x_1,x_3) = -1/4\ \varepsilon x_1^3 + 3/4\ \varepsilon x_3^3 + 3/2\ \varepsilon x_1^2 x_3 + (1-9\gamma^2)x_3 = 0.$$

Assuming one knows an approximate solution $(a_1 a_3)$ (this can be a very crude approxmation), these equations can be expanded in a Taylor Series about this approximate solution

$$f(x_1,x_3) = f(a_1,a_3) + \partial f/\partial x_1\ (x_1-a_1) + \partial f/\partial x_3\ (x_3-a_3) + \ldots = 0$$

and

$$g(x_1,x_3) = g(a_1,a_3) + \partial g/\partial x_1\ (x_1-a_1) + \partial g/\partial x_3\ (x_3-a_3) + \ldots = 0$$

and by ignoring the higher order terms, these reduce to a set of linear algebraic equations which can be solved for a more precise value of the solution x_1,x_3. This process is repeated until a desired accuracy is reached.

The approximate solution used to start this procedure can often be a very poor approximation. Usually, this simply requires additional iterations to converge on the precise value. Sometimes, however, the process fails to converge. This doesn't present a real problem in that one can specify a limiting number of iterations to be performed. If convergence is not achieved within this limiting number of iterations, a new starting point is chosen and the process repeated.

Another problem occurs when the equations have mulitple sets of real roots. In Duffing's Equation, these multiple roots represent the well known jump phenomena. Since each set of solutions satisfies the equations equally as well, a criteria must be established in order to choose the appropriate solution. While further investigation is required, at this time it appears that the correct solution for our physical systems is the solution that minimizes the total system energy.

COMPARISON OF APPROXIMATE SOLUTION WITH NUMERICAL RESULTS

In order to check the accuracy of the harmonic-balance solution to Duffing's equation, numerical simulations were performed using the IBM program CSMP. Since the approximate solution doesn't consider initial conditions, the numerical simulation was continued for sufficient periods of time to ensure that all transient responses had been eliminated. The output of the CSMP simulation was used in a Fourier Analysis routine to determine the amplitude, phase and frequency characteristics of the steady-state response. The results of several analyses, along with the approximate solutions are shown as amplitude and phase plots shown in Fig. 2 through Fig. 5. Note that the approximate solutions are represented by a smooth curve, while the numerical solutions are represented by discrete points.

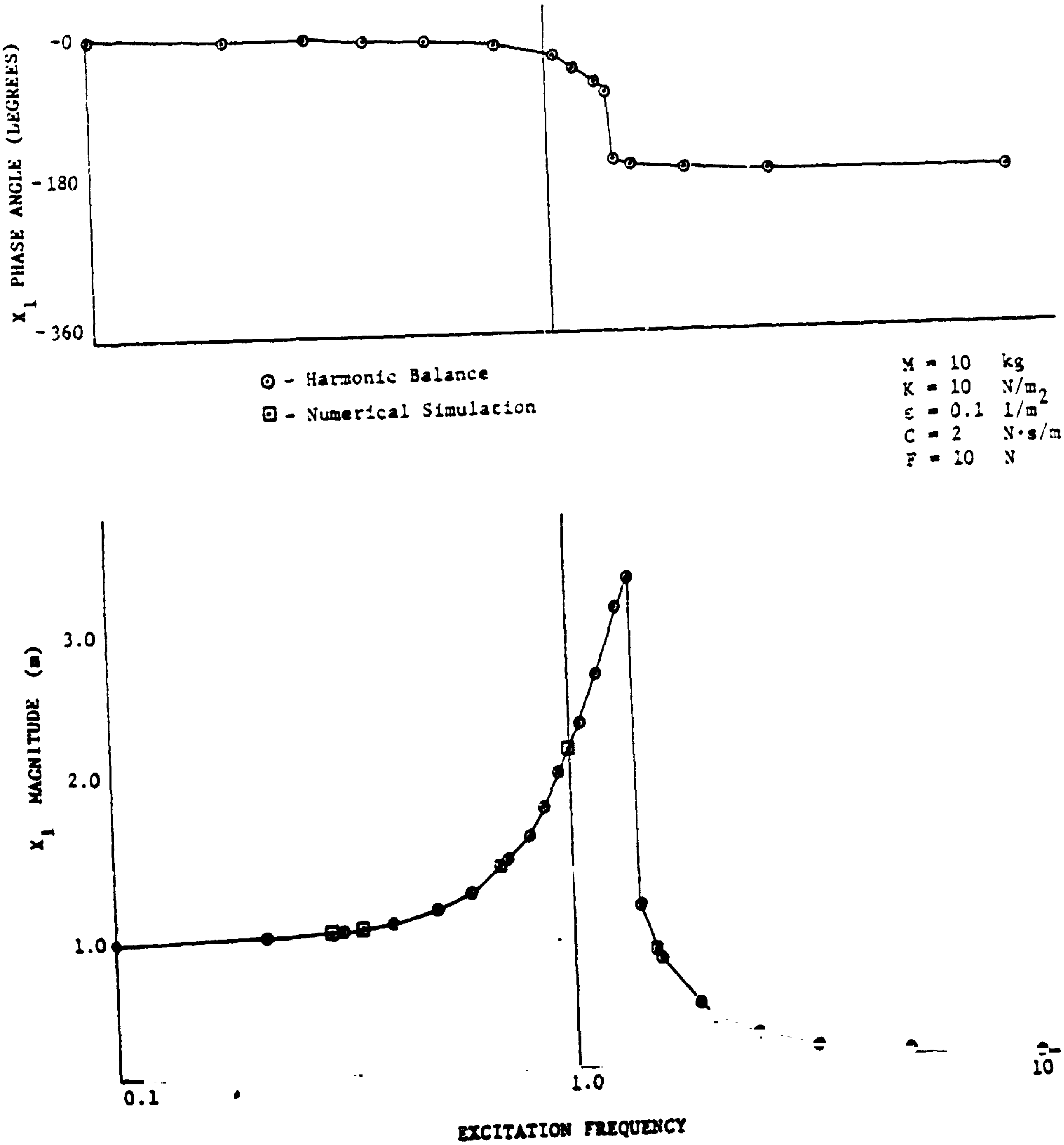


Fig. 2 Steady-State Solution of Duffing's Equation: Amplitude and Phase Relationship of the Fundamental Harmonic Component for the Case of Small Nonlinearity.

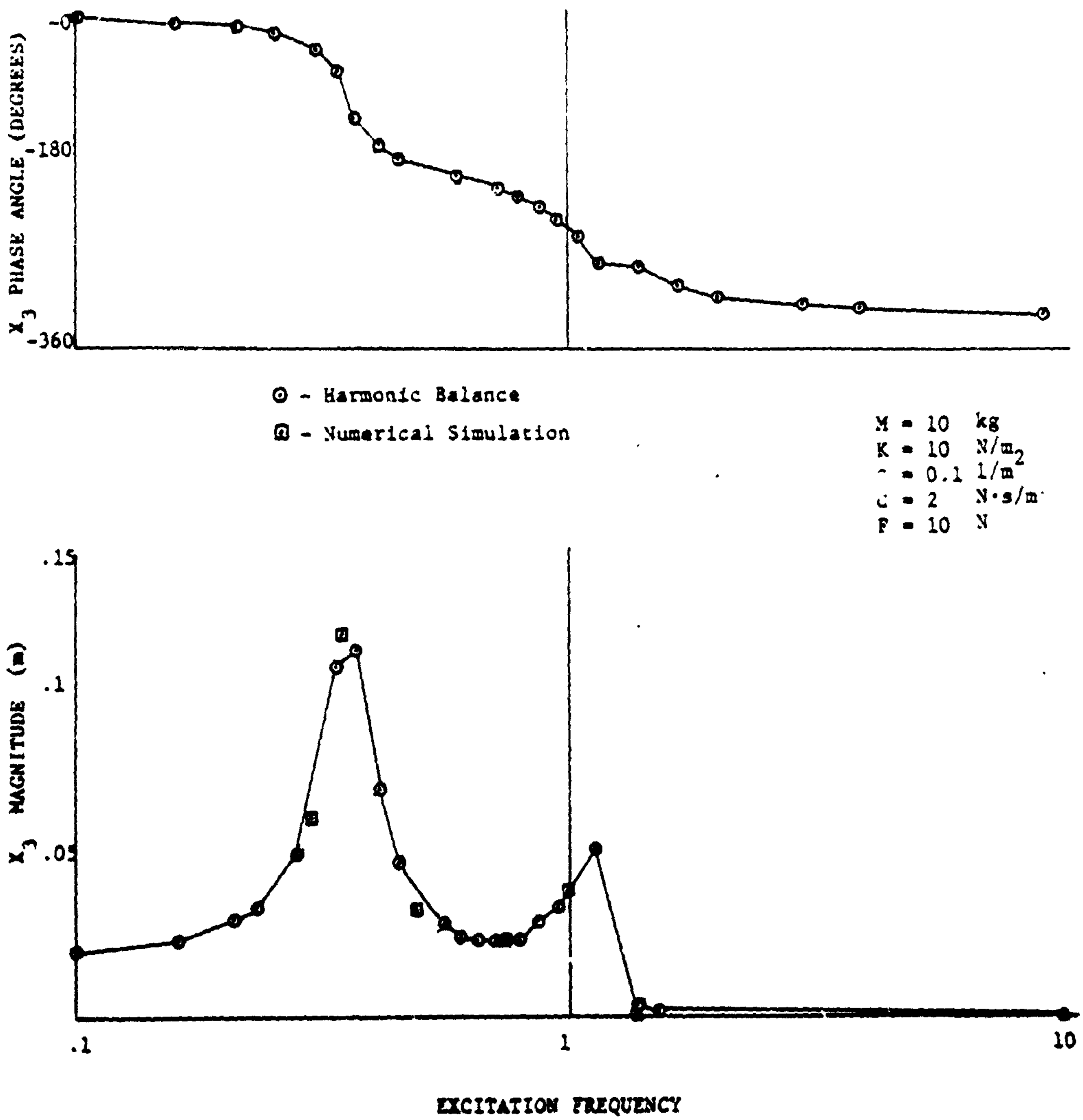


Fig. 3 Steady-State Solution of Duffing's Equation: Amplitude and Phase Relationship of the Third Harmonic Component for the Case of Small Nonlinearity.

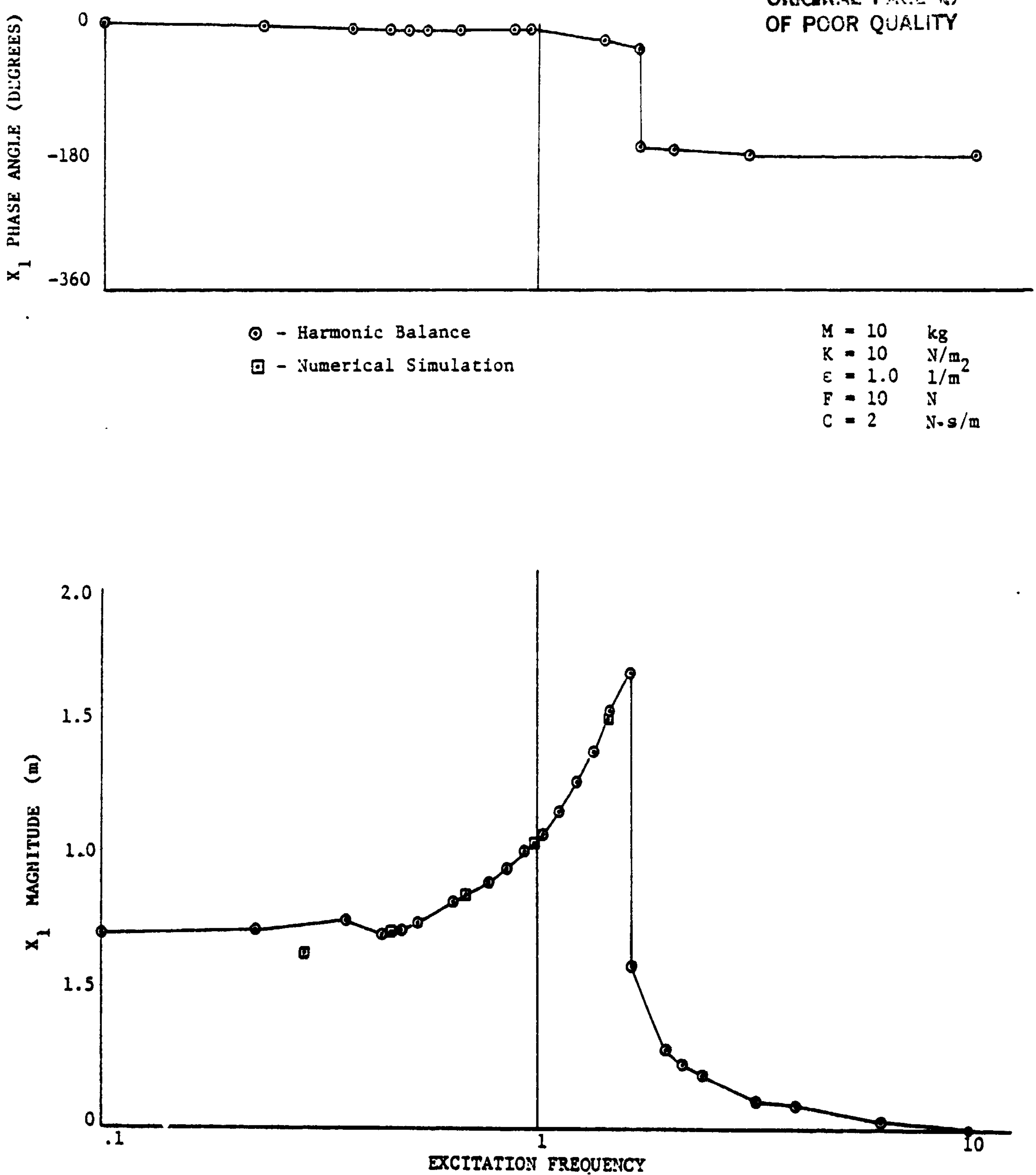


Fig. 4 Steady-State Solution of Duffing's Equation: Amplitude and Phase Relationship of the Fundamental Harmonic Component for the Case of Large Nonlinearity.

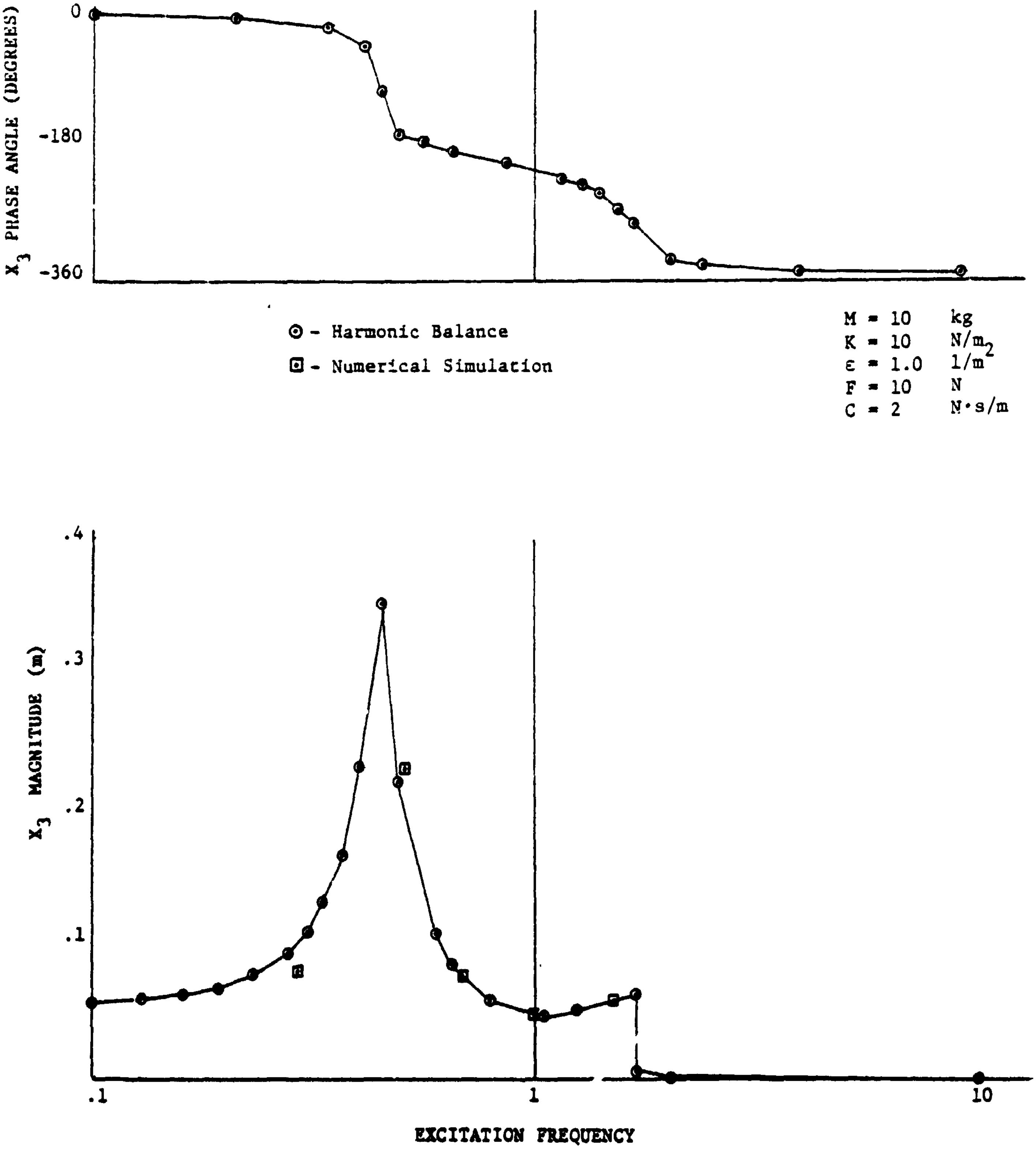


Fig. 5 Steady-State Solution of Duffing's Equation: Amplitude and Phase Relationship of the Third Harmonic Component for the Case of Large Nonlinearity.

This is due to the fact that the numerical solutions where somewhat expensive, hence only a few frequencies were chosen at which to perform the numerical simulations.

The system represented by Fig. 2 and Fig. 3 had the parameters adjusted so that the nonlinearities were small. As can be seen, the approximate and numerical solutions are in good agreement for this case. The system represented by Fig. 4 and Fig. 5 had the parameters adjusted so that the nonlinearities were large. By large, one means that the maximum amplitude of the nonlinear terms in the differential equation were of the same magnitude as that of the linear terms. As can be seen, the harmonic-balance solution produced reasonably good results in this case.

As a final check of the harmonic-balance technique, the magnitudes of the Fourier components obtained from the numerical simulations were compared. These Fourier components appear in Table 1. As can be seen from Table 1, only the first and third harmonic components are significant, the others being much smaller than these two dominant components. Hence, additional harmonic terms were not necessary in the initial assumed solution.

Hence, the results of this portion of the investigation indicate that the harmonic-balance technique can be used to produce reasonably good solutions to this type of nonlinear differential equation, even when the nonlinearities are large.

$$x(t) = \frac{X_o}{2} + \sum_{n=1}^{\infty} X_n \cos(n\omega t - \phi_n)$$

FOURIER COEFFICIENT	SMALL NONLINEARITY	LARGE NONLINEARITY
X_o	-3.3×10^{-6}	-1.0×10^{-7}
X_1	1.162	7.38×10^{-1}
X_2	2.8×10^{-6}	1.0×10^{-7}
X_3	3.61×10^{-2}	2.308×10^{-1}
X_4	1.67×10^{-6}	1.0×10^{-7}
X_5	7.67×10^{-4}	2.75×10^{-2}

TABLE 1 Comparison of Harmonic Components Obtained from the Numerical Simulations for a 0.5 Hertz Excitation Frequency

SOLUTION OF DYNAMICALLY-COUPLED NONLINEAR DIFFERENTIAL EQUATIONS

The motivation for dealing with this problem is that the rotor equations are not only nonlinear in the sense of having terms raised to integer powers, but they also have terms which contain products of state variables. In the previous section, the ability of the harmonic-balance technique to solve nonlinear equations of a system with a single degree of freedom was demonstrated. Since the rotor equations are for a system with six degrees of freedom and contain the aforementioned nonlinear terms, one must demonstrate the ability of the solution technique to solve such equations adequately.

Instead of beginning with the rotor equations, a simpler set of equations was chosen. This was to allow us to become familiar with solving these types of equations before dealing with the more difficult rotor equations. To this end, we chose to analyze the spring-pendulum system shown in Fig. 6. This system was chosen because its equations of motion contain terms similar to the rotor equations. Using Lagrange's technique, the differential equations of motion for this system can be shown to be

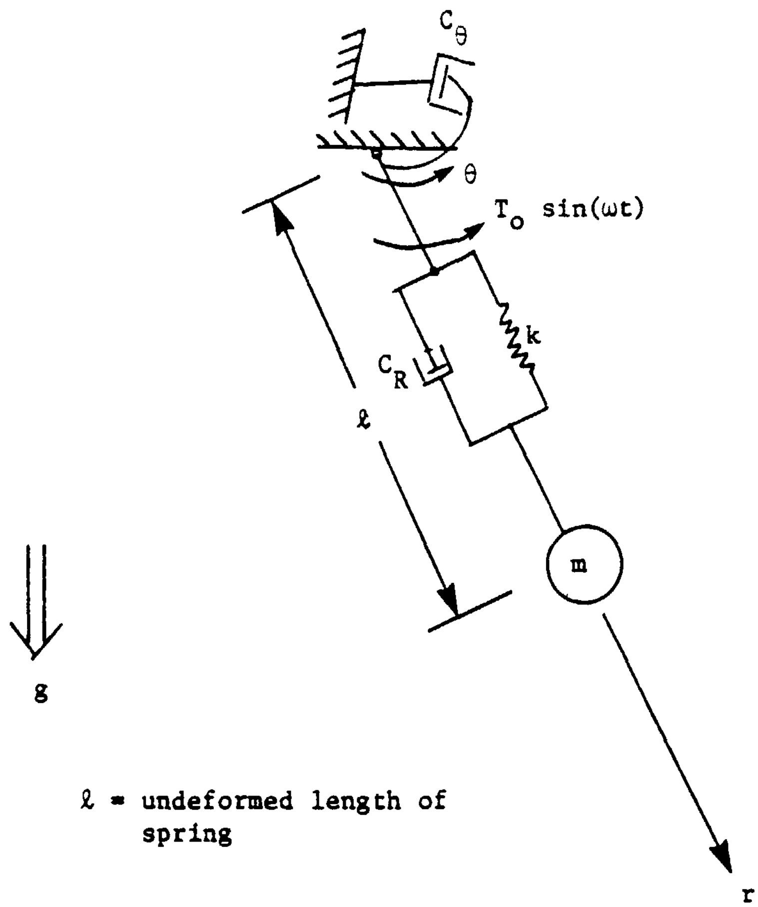
$$\ddot{r} - r\dot{\theta}^2 + C_R/m\ \dot{r} - g\cos(\theta) + \kappa/m(r-1) = 0$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + C_\theta/m\ \dot{\theta} + gr\sin(\theta) = T_o/m\ \sin(\omega t)$$

The terms of interest are

$$r\dot{\theta}^2 \text{ and } 2r\dot{r}\dot{\theta} \quad .$$

The presence of the trigonometric terms $\sin(\theta)$ and $\cos(\theta)$ pose difficulty to the solution. Since the rotor equations contain no such trigono-



ℓ = undeformed length of spring

Fig. 6 Spring-Pendulum System with Coordinate System metric terms, a truncated Taylor series was used to approximate them. The resulting equations are

$$\ddot{r} - r\dot{\theta}^2 + C_R/m\ \dot{r} - g(1-\tfrac{1}{2}\theta^2) + k/m(r-1) = 0$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} + C_\theta/m\ \dot{\theta} + gr(\theta-\theta^3/6) = T_o/m\ \sin(\omega t)$$

A comparison of the truncated Taylor series along with the appropriate trigonometric functions showed that the approximation was very good (within 10%) for angles between $\pm$ 60 degrees.

As before, since the torque is driving the system at a particular frequency, one component of the response will be at this excitation frequency. However, the presence of more than single degree of freedom can cause problems in the subsequent solution steps. This is because the harmonic-balance technique is very sensitive to the assumed form of the solution used in the equations. An assumed solution which contains erroneous harmonic terms will result in a poor solution. Hence, one must carefully consider the assumed form of the solution used.

As previously mentioned, one component of the response will be at the excitation frequency. Hence, one initially assumes

$$\theta(t) = \theta_{11}\sin(\omega t) + \theta_{12}\cos(\omega t)$$

Since there is no forcing function in the r(t) equation (the first equation), the r(t) response will be produced by gravity and the coupling term $r\dot{\theta}^2$. To gain an insight into the r(t) response, the r(t) equation can be rearranged as

$$\ddot{r} + k/m(r-1) = g(1-\tfrac{1}{2}\theta^2) + r\dot{\theta}^2.$$

This looks like the equation of a harmonic oscillator with the driving functions $g(1-\frac{1}{2}\theta^2)$ and $r\dot{\theta}^2$. The g term and the $r\dot{\theta}^2$ term will produce a constant response and a response at twice the excitation frequency. Hence, one assumes the r(t) response to be

$$r(t) = r_o + r_{21}\sin(2\omega t) + r_{22}\cos(2\omega t).$$

As before, these assumed solutions are submitted into the differential equations. By ignoring terms not in the original solution and applying the principle of harmonic balance, one produces the five nonlinear polynomials in terms of the five unknowns. The polynomials, which were found using FMACUT, are much too long to be presented here. These polynomials were then used in a solution scheme identical to that employed in solving Duffing's equation.

Again, the results of the analysis are best shown in terms of response graphs. In Fig. 7 through Fig. 9 are the amplitude and phase angle for the system of Fig. 6, with system parameters as shown on the figures. These system parameters were such that the system reponse was fairly large. In light of the nonlinearity in the equations, the agreement between the harmonic-balance and numerical solutions is considered good.

These results indicate that the harmonic balance method is indeed capable of producing good approximate solutions to coupled nonlinear differential equations of this form. The major difficulty is choosing the appropriate harmonic terms to be used in the solution. This is overcome by a careful examination of the equations and verification by

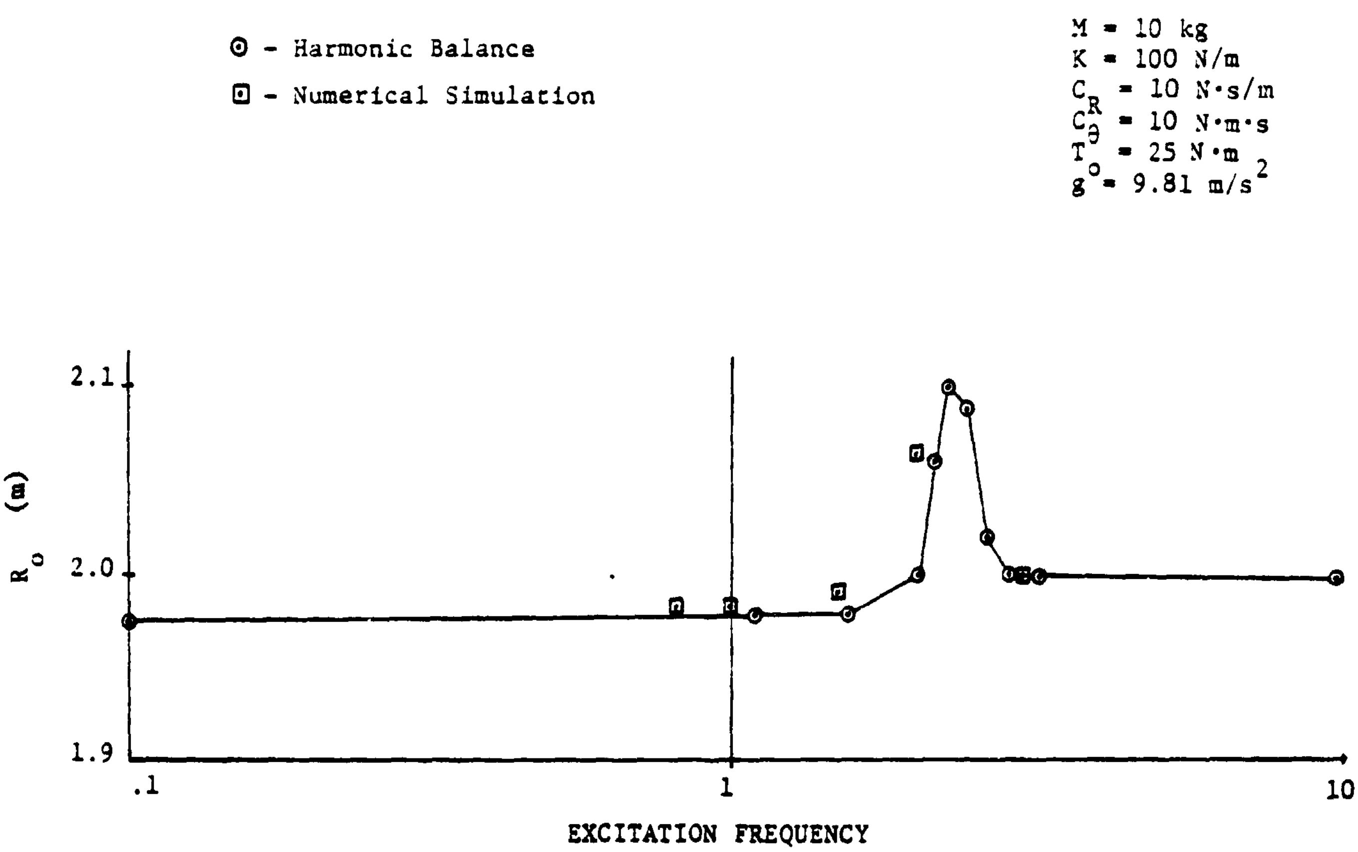


Fig. 7 **Steady-State Approximate Solution to the Spring-Pendulum Equations: R_o Response**

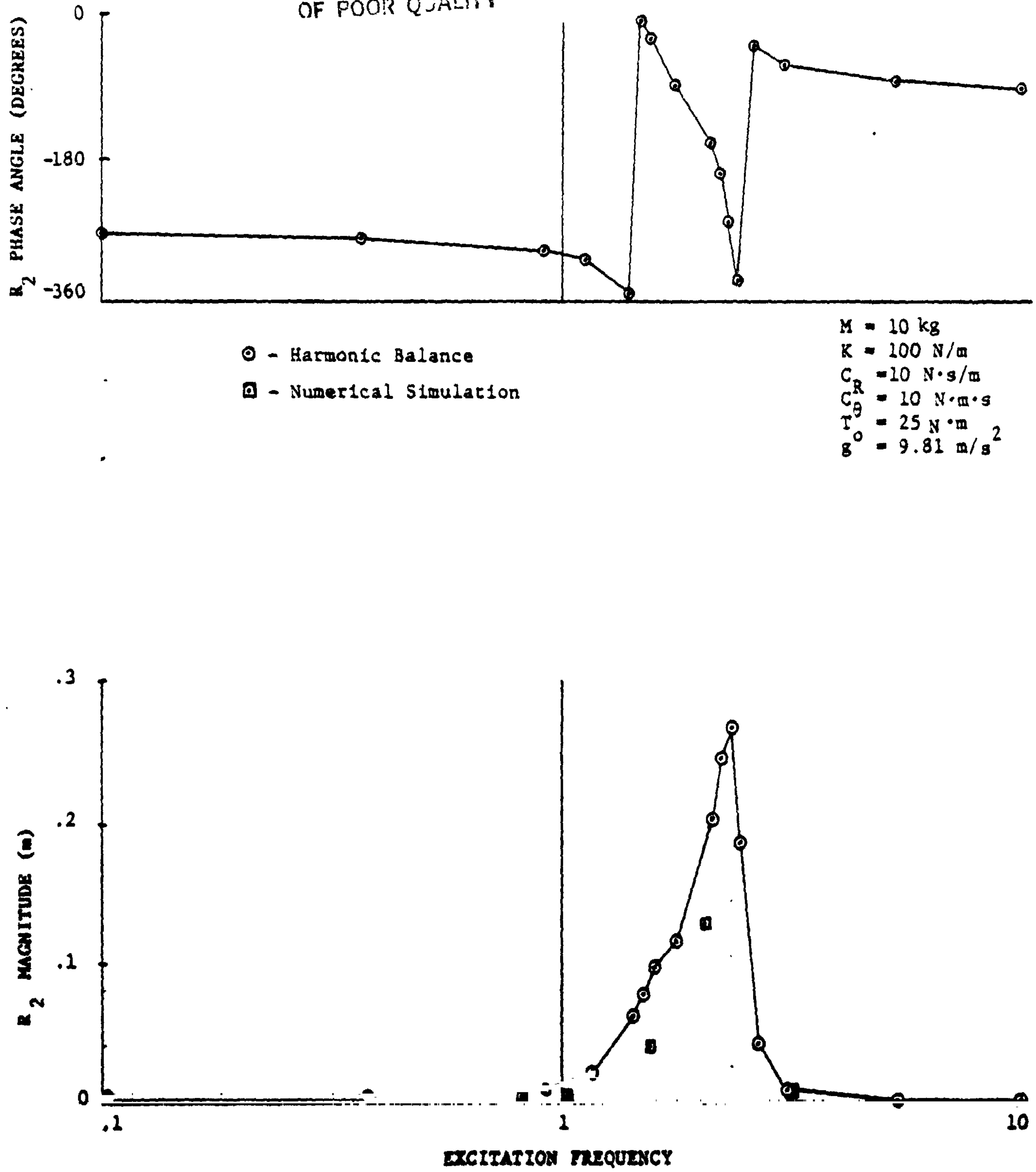


Fig. 8: Steady-State Approximate Solution to the Spring-Pendulum Equations: R_2 Response

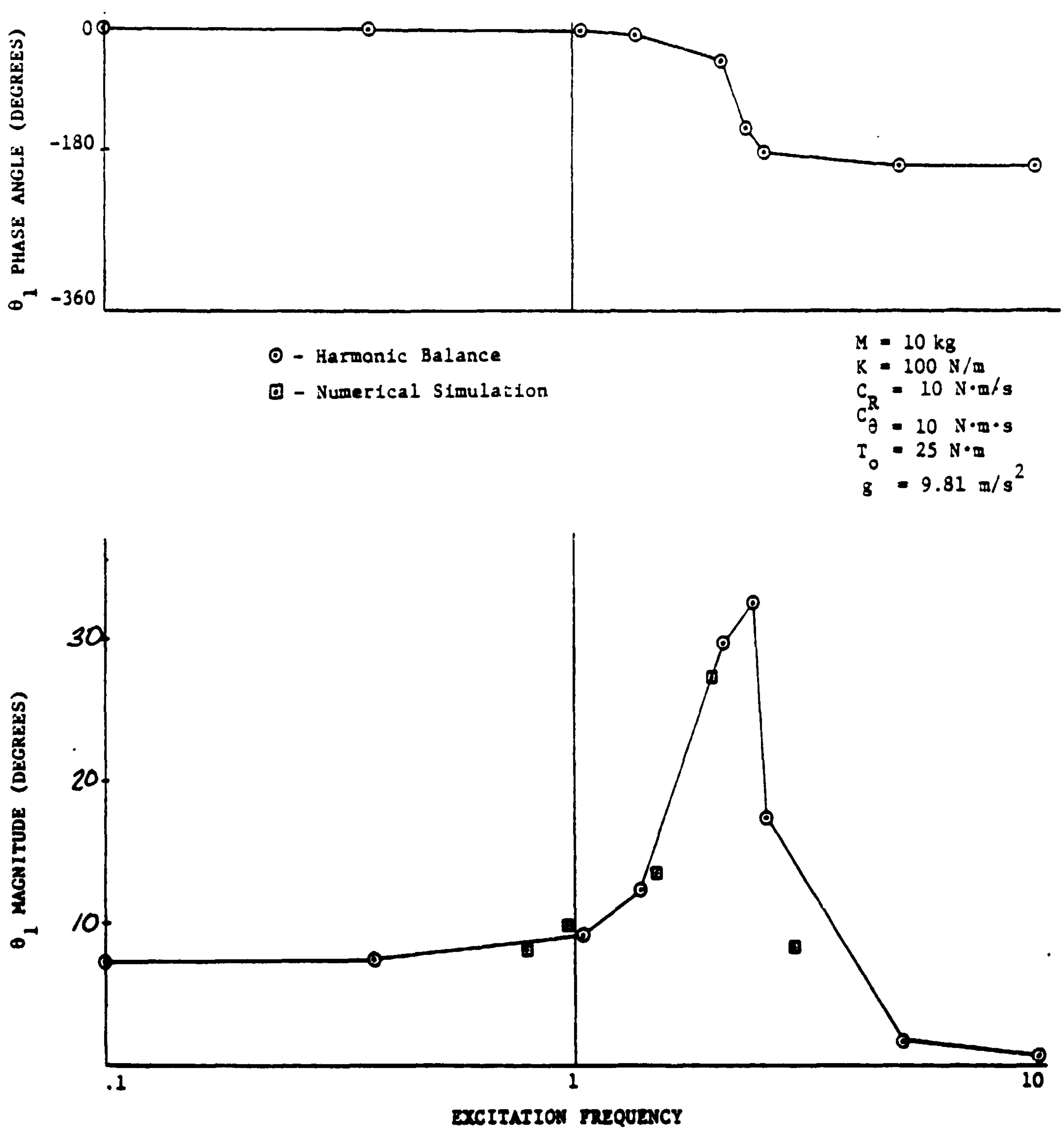


Fig. 9 Steady-State Approximate Solution to the Spring-Pendulum Equations: θ response

numerical analysis. Also, a good solution was produced using only the first terms in the assumed solution; i.e., additional iterations on the equations to get higher harmonic components were unnecessary. This is promising in that perhaps the solution of the rotor equations will not greatly expand the already large transfer matrices.

EXTENSION OF TRANSFER MATRIX METHODOLOGY TO SOLVE NONLINEAR PROBLEMS

Traditionally, transfer matrix methodology has been restricted to use in linear system analysis. However, when coupled with the harmonic-balance technique previously mentioned, transfer matrices can be used to solve nonlinear problems in an approximate fashion. The major difference is that the solution process becomes iterative for the nonlinear problem. To demonstrate the methodology, an example using springs with nonlinear stiffnesses and lumped masses is presented. The matrices can then be used to model a system represented by Duffing's equation, which provides a check on the accuracy of the technique.

As with conventional transfer matrix methods, one begins by developing the matrices for each individual element. Hence, we begin by developing the matrix for a spring having a force-deflection relationship of the form:

$$F = k(x+\epsilon x^3)$$

where x is the net spring deflection.

The free-body diagram of the spring is shown in Fig. 14. The internal force is assumed to have two frequency components; i.e.,

$$N(t) = N_1 \sin(\omega t) + N_3 \sin(3\omega t) \ .$$

This assumption results from the cubic nonlinearity in the spring. Using the force-deflection relationship and assuming a response also of the form:

$$x(t) = x_1 \sin(\omega t) + x_3 \sin(3\omega t)$$

yields the following relationship:

$$N(t) = k(x_1 \sin(\omega t) + x_3 \sin(3\omega t) - \varepsilon(x_1 \sin(\omega t) + x_3 \sin(3\omega t))^3) .$$

Simplifying and applying the principle of harmonic balance to this equation produces the following simultaneous equations:

$$N_1/k = x_1 + \varepsilon(3/4\ x_1^3 - 3/4\ x_1^2 x_3 + 3/2\ x_1 x_3^2)$$

and

$$N_3/k = x_3 + \varepsilon(-1/4\ x_1^3 + 3/2\ x_1^2 x_3 + 3/4\ x_3^3) \ .$$

To put these equations into transfer-matrix form, let

$$x_1 = x_1^R - x_1^L$$

$$x_3 = x_3^R - x_3^L \ .$$

Defining the state vector as $[x_1 \ \ N_1 \ \ x_3 \ \ N_3 \ \ | \ \ 1]^T$ these equations can be written in matrix form as

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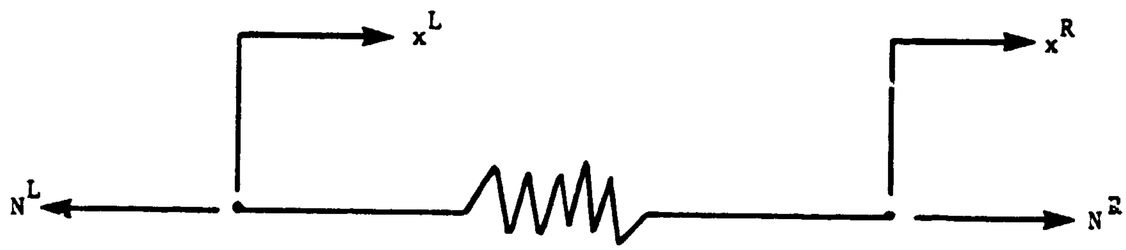


$$N^R = N^L = N_{spring}$$

$$N_{spring} = k\{(x^R - x^L) + \varepsilon(x^R - x^L)^3\}$$

Fig. 10 Free-Body Diagram of Nonlinear Spring

$$\begin{Bmatrix} x_1 \\ N_1 \\ x_3 \\ N_3 \\ \hline 1 \end{Bmatrix}^R = \left[\begin{array}{cccc|c} 1 & 1/k & 0 & 0 & NL_1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/k & NL_3 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{Bmatrix} x_1 \\ N_1 \\ x_3 \\ N_3 \\ 1 \end{Bmatrix}^L$$

where

$$NL_1 = -\varepsilon(3/4\ x_1{}^3 - 3/4\ x_1{}^2 x_3 + 3/2\ x_1 x_3{}^2)$$

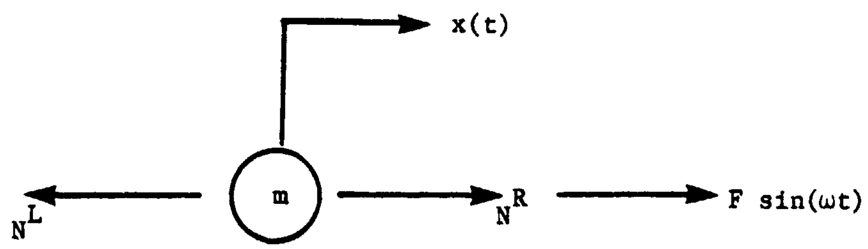
$$NL_3 = -\varepsilon(-1/4\ x_1{}^3 + 3/2\ x_1{}^2 x_3 + 3/4\ x_3{}^3)$$

Note that the nonlinear terms are not separated into left and right variables, but simply carried along in the extension column as a correction term. This transfer matrix is best understood as a set of nonlinear polynomials to which a solution must be found, in cont ast to a standard transfer matrix which represents a set of simultaneous linear equations which can be explicitly solved. Hence, the solution using this matrix in effect seeks values of the system variables which satisfy the equations and the boundary conditions.

To develop the transfer matrix for a lumped mass, a similiar procedure is followed. The free-body diagram for the lumped mass is shown in Fig. 15. Note the assumed form of the response is consistent with that of the spring. Since this is a point transfer matrix;

$$x_1{}^R = x_1{}^L$$

$$x_3{}^R = x_3{}^L$$



$$x^R = x^L = x(t) = x_1 \sin(\omega t) + x_3 \sin(3\omega t)$$

$$\overset{+}{\rightarrow} \Sigma F = m\ddot{x} \rightarrow N^R - N^L + F = -m\omega^2 \{\sin(\omega t) + 9 \sin(3\omega t)\}$$

Fig. 11 Free-Body Diagram of Point Mass

Applying Newton's second law of motion and the principle of harmonic balance yields the other equations

$$N_1^R = N_1^L - m\omega^2 x_1 - F$$

$$N_3^2 = N_3^L - 9\, m\omega^2 x_3.$$

Note the inclusion of the external excitation $F\sin(\omega t)$. These can now be written in matrix form as

$$\begin{Bmatrix} x_1 \\ N_1 \\ x_3 \\ N_3 \\ \hline 1 \end{Bmatrix} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -m\omega^2 & 1 & 0 & 0 & -F \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -9m\omega^2 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{Bmatrix} x_1 \\ N_1 \\ x_3 \\ N_3 \\ \hline 1 \end{Bmatrix}^L$$

Hence, systems with these types of elements can be modeled by appropriate multiplication of transfer matrices and application of boundary conditions. However, one problem is quite noticeable. That is, the nonlinear terms in the extension column of the spring transfer matrix preclude an explicit solution. This problem is overcome in the same fashion as finding the roots of the nonlinear polynomials previously discussed.

The solution is initially formulated by ignoring the nonlinear terms in the extension column. Once an initial solution is found, this solution is used to predict the magnitude of the nonlinear terms in the matrix extension column. This prediction doesn't use the linear solution directly, but predicts different values by using a truncated Taylor Series. The solution is then reformulated, including the nonlinear correction terms, and a new solution is found. This process is repeated until a desired accuracy is achieved. Again, we will use an example to help illustrate the method.

For the system shown in Fig. 1, the boundary conditions are

$$x_1^L = x_3^R = 0; \quad N_1^R = N_3^R = 0.$$

Hence, the initial solution is found by ignoring the nonlinear correction terms in the spring transfer matrix, which results in the linear solution

$$x_1 = F/k(1/(1-\gamma^2))$$

$$\gamma^2 = \frac{\omega^2 m}{K}$$

and $x_3 = 0$. Recall that the previous equations developed for the spring are

$$x_1+\varepsilon(3/4\ x_1^3-3/4\ x_1^2x_3+3/2\ x_1x_3^2)-N_1/k = 0$$

$$x_3+\varepsilon(-1/4\ x_1^3+3/2\ x_1^2x_3+3/4\ x_3^3)-N_3/k = 0$$

These equations can be regarded as two simultaneous equations for which a solution is desired. Using the above linear solution as an approximate solution to these equations, one can expand these equations in a truncated Taylor series yielding two simultaneous linear algebraic equations which can be solved for a more precise solution. The resulting solution to these equations can then be used to more accurately predict the nonlinear correction terms in the spring transfer matrix. This provides for better convergence of the overall solution. The overall solution is then reformulated, using the nonlinear correction terms in the spring transfer matrix, and a new solution found. This process is repeated until a desired accuracy is achieved.

In regions where multiple solutions are possible, i.e., as with the jump phenomena, one can find these multiple solutions. This is accomplished by resolving the problem using a different initial solution; i.e., one that differs from the linear solution. This is done by modifying the initial state vector in the initial solution. This change is somewhat arbitrary in that it only provides a starting point from which to begin the solution process.

The results obtained (for the system of Fig. 1) with this procedure were identical to those obtained using the harmonic-balance principle on the differential equation. In obtaining the numerical value of the solution, this method required twice the number of iterations as the harmonic-balance method. This is because the transfer matrix method calculates displacement and internal force instead of just displacement, as in the harmonic-balance method.

SUMMARY

This report has outlined what is believed to be very significant work in the area of the advancement of transfer matrix methodology. Problems with large and small nonlinearities have been solved with the proposed harmonic-balance technique. This technique was subsequently implemented in a transfer matrix solution scheme which allows the solu-

tion of nonlinear structural mechanics problems via a transfer matrix scheme.

This work has put the project in the position that provides confidence that its implementation on a geared rotor analysis can proceed in an orderly and successful manner.